A vertical Liouville subfoliation on the cotangent bundle of a Cartan space and some related structures

Cristian Ida and Adelina Manea

Abstract

In this paper we study some problems related to a vertical Liouville distribution on the cotangent bundle of a Cartan space. We study the cohomology of c–indicatrix cotangent bundle and the existence of some linear conections of Vrănceanu and Vaisman type on Cartan spaces related to foliated structures. Also we identify a certain $(n, 2n-1)$–codimensional subfoliation $(\mathcal{F}_V, \mathcal{F}_{C^*})$ on T^*M_0 given by vertical foliation $\mathcal{F}_V$ and the line foliation $\mathcal{F}_{C^*}$ spanned by the vertical Liouville-Hamilton vector field C^* and we give a triplet of basic connections adapted to this subfoliation.

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1 Introduction and preliminaries

1.1 Introduction

The study of interrelations between the geometry of foliations on the tangent manifold of a Finsler space and the geometry of the Finsler space itself was initiated and intensively studied by Bejancu and Farran [9]. The main idea of their paper is to emphasize the importance of some foliations which exist on the tangent bundle of a Finsler space (M, F), in studying the differential geometry of (M, F) itself. On the other hand in a very recent paper [3] a similar study of some natural foliations in cotangent bundle T^*M of a Cartan space (M, K) is given. It is shown that geometry of these foliations is closely related to the geometry of the Cartan space (M, K) itself. This approach is used to obtain new characterizations of Cartan spaces with negative constant curvature.

The aim of our paper is to continue the study of these foliations on the cotangent bundle of a Cartan space (M, K) from more points of view as: the cohomology of c–indicatrix cotangent bundle, the study of some linear conections of Vrănceanu and Vaisman type on Cartan spaces related to foliated structures, the study of a certain $(n, 2n-1)$–codimensional subfoliation $(\mathcal{F}_V, \mathcal{F}_{C^*})$ on T^*M_0 given by vertical foliation $\mathcal{F}_V$ and the line foliation $\mathcal{F}_{C^*}$ spanned by the vertical Liouville-Hamilton vector field C^* and the existence

of a triplet basic connections adapted to this subfoliation. The notions are introduced by analogy with similar notions on the tangent manifold of a Finsler space.

The paper is organized as follows: In the preliminary subsection we briefly recall some basic facts on the geometry of a Cartan space (M, K) and we present the almost Kählerian model (T^*M_0, G, J) of the cotangent manifold $T^*M_0 = T^*M - \{zero\ section\}$ together with the Riemannian metric G given by Sasaki type lift of the fundamental metric tensor $g^{ij} = \frac{1}{2}\frac{\partial^2 K^2}{\partial p_i \partial p_j}$ and with the natural almost complex structure J. In the second section, following an argument inspired from [7], we define a vertical Liouville distribution $\mathcal{L}_{C^*}$ on T^*M_0 as the complementary orthogonal distribution in the vertical distribution $V(T^*M_0)$ to the line distribution spanned by the vertical Liouville-Hamilton vector field $C^* = p_i\frac{\partial}{\partial p_i}$ and we prove that the distribution $\mathcal{L}_{C^*}$ is an integrable one. Also, we find geometric properties of both leaves of vertical Liouville distribution $\mathcal{L}_{C^*}$ and the vertical distribution $V(T^*M_0)$. We notice that the vertical Liouville distribution $\mathcal{L}_{C^*}$ will be an important tool in study of future problems in this paper. In the third section, using the vertical Liouville-Hamilton vector field C^* and the natural almost complex structure J on T^*M_0, we give an adapted basis in $T(T^*M_0)$. Next we prove that the c–indicatrix cotangent bundle $I(M, K)(c)$ of (M, K) is a CR–submanifold of the almost Kählerian manifold (T^*M_0, G, J) and we study some comological properties of $I(M, K)(c)$ in relation with classical cohomology of CR-submanifolds, [13]. In the fourth section, following some ideas from [8], we investigate the existence of some linear connections of Vrănceanu and Vaisman type on a Cartan space, related with the vertical and Liouville-Hamilton foliations on it. In the last section, following [15], we briefly recall the notion of a (q_1, q_2)-codimensional subfoliation on a manifold and we identify a $(n, 2n-1)$-codimensional subfoliation $(\mathcal{F}_V, \mathcal{F}_{C^*})$ on the cotangent manifold T^*M_0 of a Cartan space (M, K), where $\mathcal{F}_V$ is the vertical foliation and $\mathcal{F}_{C^*}$ is the line foliation spanned by the vertical Liouville-Hamilton vector field C^*. Firstly we make a general approach about basic connections on the normal bundles related to this subfoliation and next a triple of adapted basic connections with respect to this subfoliation is given. A similar study on the tangent manifold of a Finsler space (M, F) is given in [18].

1.2 Preliminaries and notations

In this subsection we briefly recall some basic facts from the geometry of a Cartan space (M, K). For more see [19, 20, 21, 23].

Let M be an n-dimensional C^∞ manifold and $\pi^* : T^*M \to M$ its cotangent bundle. If (x^i), $i = 1, \ldots, n$ are local coordinates in a local chart U on M, then (x^i, p_i), $i = 1, \ldots, n$ will be taken as local coordinates in the local chart $(\pi^*)^{-1}(U)$ on T^*M with the momenta (p_i) provided by $p = p_i dx^i$ where $p \in T^*_xM$, $x \in M$ and $\{dx^i\}$ is the natural basis of T^*_xM. The indices $i, j, k, \ldots$ will run from 1 to n and the Einstein convention on summation will be used. A change of coordinates on $U \cap \widetilde{U} \neq \emptyset$ given by $(x^i, p_i) \mapsto (\widetilde{x}^i, \widetilde{p}_i)$ has the form

$$\widetilde{x}^i = \widetilde{x}^i(x^i)\,,\ \widetilde{p}_i = \frac{\partial x^j}{\partial \widetilde{x}^i}p_j\,,\ rank\left(\frac{\partial \widetilde{x}^i}{\partial x^j}\right) = n, \tag{1.1}$$

where $\left(\frac{\partial x^j}{\partial \widetilde{x}^i}\right)$ is the inverse of the Jacobian matrix $\left(\frac{\partial \widetilde{x}^i}{\partial x^j}\right)$.

Let us consider $\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial p_i}\right\}$, $i = 1, \ldots, n$ the natural basis in $T_{(x,p)}(T^*M)$ and $\{dx^i, dp_i\}$ the dual basis of it. The change of coordinates (1.1) produces the following changes:

$$\frac{\partial}{\partial x^i} = \frac{\partial \widetilde{x}^j}{\partial x^i}\frac{\partial}{\partial \widetilde{x}^j} + \frac{\partial \widetilde{p}_j}{\partial x^i}\frac{\partial}{\partial \widetilde{p}_j}\,,\ \frac{\partial}{\partial \widetilde{p}_i} = \frac{\partial \widetilde{x}^i}{\partial x^j}\frac{\partial}{\partial p_j} \tag{1.2}$$

and

$$d\widetilde{x}^i = \frac{\partial \widetilde{x}^i}{\partial x^j}dx^j\ ,\ d\widetilde{p}_i = \frac{\partial x^j}{\partial \widetilde{x}^i}dp_j + \frac{\partial^2 x^j}{\partial \widetilde{x}^i \partial \widetilde{x}^k}p_j dx^k. \tag{1.3}$$

The kernel $V_{(x,p)}(T^*M)$ of the differential $d\pi^* : T_{(x,p)}(T^*M) \to T_xM$ is called the *vertical* subspace of $T_{(x,p)}(T^*M)$ and the mapping $(x,p) \mapsto V_{(x,p)}(T^*M)$ is a regular distribution on T^*M called the *vertical distribution.* This is integrable and defines the vertical foliation $\mathcal{F}_V$ with the leaves characterized by $x^k = constant$ and it is locally spanned by $\left\{\frac{\partial}{\partial p_i}\right\}$. The vector field $C^* = p_i\frac{\partial}{\partial p_i}$ is called the *vertical Liouville-Hamilton* vector field and $\omega = p_i dx^i$ s called the Liouville 1-form on T^*M. Then $\Omega = d\omega = dp_i \wedge dx^i$ is the canonical symplectic structure on T^*M. For an easer handling of the geometrical objects on T^*M it is usual to consider a supplementary distribution to the vertical distribution, $(x,p) \mapsto H_{(x,p)}(T^*M)$, called the *horizontal* distribution and to report all geometrical objects on T^*M to the decomposition

$$T_{(x,p)}(T^*M) = H_{(x,p)}(T^*M) \oplus V_{(x,p)}(T^*M). \tag{1.4}$$

The horizontal distribution is taken as being locally spanned by the local vector fields

$$\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} + N_{ij}(x,p)\frac{\partial}{\partial p_j}. \tag{1.5}$$

The horizontal distribution is called also a *nonlinear connection* on T^*M and the functions N_{ij} are called the local coefficients of this nonlinear connection. It is important to note that any regular Hamiltonian on T^*M determines a nonlinear connection whose local coefficients verify $N_{ij} = N_{ji}$. The basis $\left\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial p_i}\right\}$ is adapted to the decomposition (1.4). The dual of it is $\{dx^i, \delta p_i := dp_i - N_{ji}dx^j\}$.

According to [23] a *Cartan structure* on M is a function $K : T^*M \to [0,\infty)$ which has the following properties:

i) K is C^∞ on $T^*M_0 := T^*M - \{\text{zero section}\}$;

ii) $K(x,\lambda p) = \lambda K(x,p)$ for all $\lambda > 0$;

iii) the $n \times n$ matrix (g^{ij}), where $g^{ij} = \frac{1}{2}\frac{\partial^2 K^2}{\partial p_i \partial p_j}$, is positive definite at all points of T^*M_0.

We notice that in fact $K(x,p) > 0$, whenever $p \neq 0$.

Definition 1.1. The pair (M, K) is called a Cartan space.

Let us put

$$p^i = \frac{1}{2}\frac{\partial K^2}{\partial p_i}\,,\ C^{ijk} = -\frac{1}{4}\frac{\partial^3 K^2}{\partial p_i \partial p_j \partial p_k}. \tag{1.6}$$

The properties of K imply that

$$p^i = g^{ij} p_j\,,\ p_i = g_{ij} p^j\,,\ K^2 = g^{ij} p_i p_j = p_i p^i\,,\ C^{ijk} p_k = C^{ikj} p_k = C^{kij} p_k = 0, \tag{1.7}$$

where (g_{ij}) is the inverse matrix of (g^{ji}).

One considers the formal Christoffel symbols by

$$\gamma^i_{jk}(x,p) := \frac{1}{2} g^{is}\left(\frac{\partial g_{js}}{\partial x^k} + \frac{\partial g_{sk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^s}\right), \tag{1.8}$$

and the contractions $\gamma^0_{jk}(x,p) := \gamma^i_{jk}(x,p) p_i$, $\gamma^0_{j0} := \gamma^i_{jk} p_i p^k$. Then the functions

$$N_{ij}(x,p) = \gamma^0_{ij}(x,p) - \frac{1}{2}\gamma^0_{h0}(x,p)\frac{\partial g_{ij}}{\partial p_h}(x,p), \tag{1.9}$$

define a nonlinear connection on T^*M. This nonlinear connection was discovered by R. Miron [21] and is called the *canonical nonlinear connection* of (M, K). We also notice that the coefficients from (1.9) satisfies $N_{ij}(x,p) = N_{ji}(x,p)$ and are positively homogeneous of degree 1 in momenta. For details we refer to Ch. 6 in [23]. Thus a decomposition (1.4) holds. From now on we shall use only the canonical nonlinear connection given by (1.9).

To a Cartan space (M, K) we can associate some important geometrical object fields on the manifold T^*M_0. Namely, the N-lift G of the fundamental tensor g^{ij}, the almost complex structure J, etc. If N is the canonical nonlinear of (M, K), thus (G, J) determine an almost Hermitian structure, which is derived only from the fundamental function K of the Cartan space.

The N-lift of the fundamental tensor field g^{ij} of the space (M, K) is defined by

$$G = g_{ij} dx^i \otimes dx^j + g^{ij} \delta p_i \otimes \delta p_j \tag{1.10}$$

and there is a natural almost complex structure J on T^*M_0 which is locally given by

$$J = -g_{ij}\frac{\partial}{\partial p_i} \otimes dx^j + g^{ij}\frac{\delta}{\delta x^i} \otimes \delta p_j\,,\ J\left(\frac{\delta}{\delta x^i}\right) = -g_{ij}\frac{\partial}{\partial p_j}\,,\ J\left(\frac{\partial}{\partial p_i}\right) = g^{ij}\frac{\delta}{\delta x^j}. \tag{1.11}$$

Thus, according to [23], (T^*M_0, G, J) has a model of almost Kählerian manifold with the fundamental form Ω given by $\Omega(X, Y) = G(JX, Y)$ and locally expressed by

$$\Omega = \delta p_i \wedge dx^i = dp_i \wedge dx^i. \tag{1.12}$$

2 A vertical Liouville distribution on T^*M_0

Following an argument inspired from [7] we define a vertical Liouville distribution on T^*M_0 as the complementary orthogonal distribution in $V(T^*M_0)$ to the line distribution spanned by the vertical Liouville-Hamilton vector field C^* and we prove that this distribution is an integrable one.

By (1.7) we have

$$G(C^*, C^*) = K^2. \tag{2.1}$$

By means of G and C^*, we define the vertical one form ζ by

$$\zeta(X) = \frac{1}{K} G(X, C^*) \ \forall\, X \in \Gamma(V(T^*M_0)). \tag{2.2}$$

Denote by $\{C^*\}$ the line vector bundle over T^*M_0 spanned by C^* and define the *Liouville distribution* as the complementary orthogonal distribution $\mathcal{L}_{C^*}$ to $\{C^*\}$ in $V(T^*M_0)$ with respect to G. Hence, $\mathcal{L}_{C^*}$ is defined by ζ, that is we have

$$\Gamma\left(\mathcal{L}_{C^*}\right) = \{X \in \Gamma(V(T^*M_0)) \,:\, \zeta(X) = 0\}. \tag{2.3}$$

Thus, any vertica vector field $X = X_i \frac{\partial}{\partial p_i}$ can be expressed as follows:

$$X = PX + \frac{1}{K}\zeta(X) C^*, \tag{2.4}$$

where P is the projection morphism of $V(T^*M_0)$ on $\mathcal{L}_{C^*}$. By direct calculations, we obtain

$$G(X, PY) = G(PX, PY) = G(X, Y) - \zeta(X)\zeta(Y), \ \forall\, X, Y \in \Gamma(V(T^*M_0)). \tag{2.5}$$

Then the local components of ζ and P with respect to the basis $\{\delta p_i\}$ and $\left\{\delta p_i \otimes \frac{\partial}{\partial p_i}\right\}$, respectively, are given by

$$\zeta^i = \frac{p^i}{K}\,,\ P^i_j = \delta^i_j - \frac{\zeta^i p_j}{K}, \tag{2.6}$$

where δ^i_j are the components of the Kronecker delta.

Theorem 2.1. *The Liouville distribution $\mathcal{L}_{C^*}$ is integrable and it defines a foliation on T^*M_0 denoted by $\mathcal{F}_{C^*}$.*

Proof. Let $X, Y \in \Gamma\left(\mathcal{L}_{C^*}\right)$. As $V(T^*M_0)$ is an integrable distribution on T^*M_0, it is sufficient to prove that $[X, Y]$ has no component with respect to C^*.

It is easy to see that a vertical vector field $X = X_i \frac{\partial}{\partial p_i}$ is in $\Gamma\left(\mathcal{L}_{C^*}\right)$ if and only if

$$g^{ij} X_i p_j = 0. \tag{2.7}$$

Differentiate (2.7) with respect to p_k we get

$$\frac{\partial g^{ij}}{\partial p_k} X_i p_j + g^{ik} X_i + g^{ij} \frac{\partial X_i}{\partial p_k} p_j = 0\,,\ \forall\, k = 1, \dots, n \tag{2.8}$$

and taking into account the relation $\frac{\partial g^{ij}}{\partial p_k} p_j = 0$ (see (1.7)), one gets

$$g^{ik} X_i + g^{ij} p_j \frac{\partial X_i}{\partial p_k} = 0\,,\ \forall\, k = 1, \dots, n. \tag{2.9}$$

Then, by direct calculations using (2.9), we have

$$\begin{aligned} G([X,Y], C^*) &= g^{ij} p_j \left(\frac{\partial Y_i}{\partial p_k} X_k - \frac{\partial X_i}{\partial p_k} Y_k \right) \\ &= -g^{ik} Y_i X_k + g^{ik} X_i Y_k \\ &= 0 \end{aligned}$$

which completes the proof. □

Based on the above results, we may say that the geometry of the leaves of $\mathcal{F}_V$ should be derived from the geometry of the leaves of $\mathcal{F}_{C^*}$ and of integral curves of C^*. In order to get this interplay, we consider a leaf F_V of $\mathcal{F}_V$ given locally by $x^i = a^i$, $i = 1, \dots, n$, where the a^i's are constants. Then, $g^{ij}(a,p)$ are the components of a Riemannian metric G_{F_V} on F_V. Denote by ∇ the Levi-Civita connection on F with respect to G_{F_V} and consider the Christoffel symbols C_i^{jk} of ∇. Then we obtain the usual formula for C_i^{jk}, namely

$$C_i^{jk}(a,p) = -\frac{1}{2} g_{is}(a,p) \frac{\partial g^{sk}}{\partial p_j}(a,p) = g_{is}(a,p) C^{sjk}(a,p), \tag{2.10}$$

where $g_{is}(a,p)$ are the entries of the inverse matrix of the $n \times n$ matrix $\big(g^{si}(a,p)\big)$. Contracting (2.10) by p_j , we deduce that

$$C_i^{jk}(a,p) p_j = 0. \tag{2.11}$$

By straightforward calculations using (2.11), (2.5) and (2.6), we obtain the covariant derivatives of C^*, ζ and P in the following lemma:

Lemma 2.1. *Let (M,K) be a Cartan space. Then, on any leaf F_V of $\mathcal{F}_V$, we have*

$$\nabla_X \left(\frac{1}{K} C^* \right) = \frac{1}{K} PX, \tag{2.12}$$

$$(\nabla_X \zeta)\, Y = \frac{1}{K} G(PX, PY), \tag{2.13}$$

and

$$(\nabla_X P)\, Y = -\frac{1}{K^2} \left[G(PX,PY) C^* + K \zeta(Y) PX \right] \tag{2.14}$$

for any $X, Y \in \Gamma\,(TF_V)$.

Proof. Indeed, if we take $X = X_i \frac{\partial}{\partial p_i}$, $Y = Y_j \frac{\partial}{\partial p_j} \in \Gamma(TF_V)$ the relation (2.12) it follows by:

$$\begin{aligned}
\nabla_X \left(\frac{1}{K} C^* \right) &= \frac{X_i}{K^2} \left(\delta^i_j K - p_j \frac{\partial K}{\partial p_i} \right) \frac{\partial}{\partial p_j} + \frac{X_i p_j}{K} C^{ij}_k \frac{\partial}{\partial p_k} \\
&= \frac{X_i}{K} \left(\delta^i_j - \frac{p_j \zeta^i}{K} \right) \frac{\partial}{\partial p_j} + 0 \\
&= \frac{1}{K} PX.
\end{aligned}$$

For the relation (2.13) we have

$$(\nabla_X \zeta) Y = X(\zeta(Y)) - \zeta(\nabla_X Y) = X_i Y_j \frac{\partial \zeta^j}{\partial p_i} = \frac{X_i Y_j}{K} \left(g^{ij} - \frac{p^j p^i}{K^2} \right)$$

and

$$\frac{1}{K} G(PX, PY) = \frac{X_i Y_j}{K} \left(g^{ij} - \frac{p^j p^i}{K^2} \right).$$

The relation (2.14) it folows using (2.12) and (2.13). Indeed, we have

$$\begin{aligned}
(\nabla_X P) Y &= \nabla_X (PY) - P(\nabla_X Y) \\
&= \nabla_X \left(Y - \zeta(Y) \frac{1}{K} C^* \right) - \nabla_X Y + \frac{1}{K} \zeta(\nabla_X Y) C^* \\
&= -X(\zeta(Y)) \frac{1}{K} C^* - \zeta(Y) \frac{1}{K} PX + \frac{1}{K} \zeta(\nabla_X Y) C^* \\
&= -\left[X(\zeta(Y)) - \zeta(\nabla_X Y) \right] \frac{1}{K} C^* - \zeta(Y) \frac{1}{K} PX \\
&= -\frac{1}{K^2} \left[G(PX, PY) C^* + K \zeta(Y) PX \right].
\end{aligned}$$

□

Now, in similar manner with [7], we obtain:

Theorem 2.2. *Let (M, K) be an n-dimensional Cartan space and F_V, F_{C^*} and γ be a leaf of $\mathcal{F}_V$, a leaf of $\mathcal{F}_{C^*}$ that lies in F_V, and an integral curve of $\frac{1}{K} C^*$, respectively. Then we have the following assertions:*

i) *γ is a geodesic of F_V with respect to ∇.*

ii) *F_{C^*} is totally umbilical immersed in F_V.*

iii) *F_{C^*} lies in the indicatrix $I_a(M, K) = \{p \in T^*_a M_0 \, : \, K(a, p) = 1\}$ of (M, K) and has constant mean curvature equal to -1.*

Theorem 2.3. *Let (M, K) be an n-dimensional Cartan space and F_V be a leaf of the vertical foliation $\mathcal{F}_V$. Then the sectional curvature of any nondegenerate plane section on F_V containing the vertical Liouville-Hamilton vector field is equal to zero.*

Corollary 2.1. *Let (M, K) be an n-dimensional Cartan space. Then there exist no leaves of $\mathcal{F}_V$ which are positively or negatively curved.*

3 An adapted basis in $T(T^*M_0)$ and cohomology of the c-indicatrix cotangent bundle

In this section, using the vertical Liouville-Hamilton vector field C^* and the natural almost complex structure J on T^*M_0, we give an adapted basis in $T(T^*M_0)$. Next we prove that the c-indicatrix cotangent bundle $I(M, K)(c)$ of (M, K) is a CR-submanifold of the almost Kählerian manifold (T^*M_0, G, J) and we study some comological properties of $I(M, K)(c)$ in relation with classical cohomology of CR-submanifolds, [13].

3.1 An adapted basis in $T(T^*M_0)$

As we already saw, the vertical bundle $V(T^*M_0)$ is locally spanned by $\left\{\frac{\partial}{\partial p_i}\right\}$, $i = 1, \ldots, n$ and it admits decomposition

$$V(T^*M_0) = \mathcal{L}_{C^*} \oplus \{C^*\}. \tag{3.1}$$

In the sequel, following [17], we give another basis on $V(T^*M_0)$, adapted to $\mathcal{F}_{C^*}$, and next we extend this basis to an adapted basis in $T(T^*M^0)$, by the same reasons as in [2], [11].

We consider the following vertical vector fields:

$$\frac{\overline{\partial}}{\overline{\partial} p_j} = \frac{\partial}{\partial p_j} - t^j C^*,\ j = 1, \ldots, n, \tag{3.2}$$

where functions t^j are defined by the conditions

$$G\left(\frac{\overline{\partial}}{\overline{\partial} p_j}, C^*\right) = 0, \forall\, j = 1, \ldots, n. \tag{3.3}$$

The above conditions become

$$G\left(\frac{\partial}{\partial p_j}, p_i \frac{\partial}{\partial p_i}\right) - t^j G(C^*, C^*) = 0$$

so, taking into account also (1.10) and (2.1), we obtain the local expression of functions t^j in a local chart $(U, (x^i, p_i))$:

$$t^j = \frac{p_i g^{ji}}{K^2} = \frac{p^j}{K^2} = \frac{1}{K}\frac{\partial K}{\partial p_j} = \frac{\zeta^j}{K}, \forall\, j = 1, \ldots, n. \tag{3.4}$$

If $(\widetilde{U}, (\widetilde{x}^i, \widetilde{p}_i))$ is another local chart on T^*M_0, in $U \cap \widetilde{U} \neq \emptyset$, then we have:

$$\widetilde{t}^k = \frac{\widetilde{p}_i \widetilde{g}^{ki}}{K^2} = \frac{1}{K^2} \frac{\partial x^j}{\partial \widetilde{x}^i} p_j \frac{\partial \widetilde{x}^k}{\partial x^l} \frac{\partial \widetilde{x}^i}{\partial x^h} g^{lh} = \frac{\partial \widetilde{x}^k}{\partial x^l} t^l.$$

So, we obtain the following changing rule for the vector fields (3.2):

$$\frac{\overline{\partial}}{\overline{\partial} \widetilde{p}_i} = \frac{\partial \widetilde{x}^i}{\partial x^k} \frac{\overline{\partial}}{\overline{\partial} p_k}, \forall\, i = 1, \ldots, n. \tag{3.5}$$

By a straightforward calculation, using (1.7), it results:

Proposition 3.1. *The functions* $\{t^j\}$, $j = 1, \ldots, n$ *defined by* (3.4) *are satisfying:*

$$p_i t^i = 1\,,\; p_i \frac{\overline{\partial}}{\overline{\partial} p_i} = 0\,,\; \frac{\partial t^i}{\partial p_j} = -2 t^i t^j + \frac{g^{ij}}{K^2},\; C^* t^j = -t^j,\; \forall\, i, j = 1, \ldots, n. \tag{3.6}$$

Proposition 3.2. *The following relations hold:*

$$\left[\frac{\overline{\partial}}{\overline{\partial} p_i}, \frac{\overline{\partial}}{\overline{\partial} p_j}\right] = t^i \frac{\overline{\partial}}{\overline{\partial} p_j} - t^j \frac{\overline{\partial}}{\overline{\partial} p_i},\;\; \left[\frac{\overline{\partial}}{\overline{\partial} p_i}, C^*\right] = \frac{\overline{\partial}}{\overline{\partial} p_i}, \tag{3.7}$$

for all $i, j = 1, \ldots, n$.

By conditions (3.3), the vector fields $\left\{\frac{\overline{\partial}}{\overline{\partial} p_1}, \ldots, \frac{\overline{\partial}}{\overline{\partial} p_n}\right\}$ are orthogonal to C^*, so they belong to the $(n-1)$-dimensional distribution $\mathcal{L}_{C^*}$. It results that they are linear dependent and, from the first relation of (3.6), we have

$$\frac{\overline{\partial}}{\overline{\partial} p_n} = -\frac{p_a}{p_n} \frac{\overline{\partial}}{\overline{\partial} p_a}, \tag{3.8}$$

since the local coordinate p_n is nonzero everywhere.

We also have

Proposition 3.3. *The system* $\left\{\frac{\overline{\partial}}{\overline{\partial} p_1}, \ldots, \frac{\overline{\partial}}{\overline{\partial} p_{n-1}}, C^*\right\}$ *of vertical vector fields is a locally adapted basis to the vertical Liouville foliation* $\mathcal{F}_{C^*}$*, on* $V(T^*M_0)$.

Thus, we can denote

$$\frac{\overline{\partial}}{\overline{\partial} p_a} = E^a_i \frac{\partial}{\partial p_i},\; a = 1, \ldots, n-1,$$

where $rank\, E^a_i = n - 1$ and $E^a_i p_j g^{ij} = 0$.

Now, using the natural almost complex structure J on $T(T^*M_0)$, the new local vector field frame in $T(T^*M_0)$ is

$$\left\{\overline{X}^a, \xi^*, \frac{\overline{\partial}}{\overline{\partial} p_a}, C^*\right\}, \tag{3.9}$$

where

$$\xi^* = J\,(C^*) = p^i \frac{\delta}{\delta x^i}\,,\; \overline{X}^a = J\left(\frac{\overline{\partial}}{\overline{\partial} p_a}\right) = E^a_i g^{ij} \frac{\delta}{\delta x^j}.$$

Remark 3.1. In some future calculations we shall replace the local basis $\left\{\frac{\overline{\partial}}{\partial p_a}\right\}$, $a = 1, \ldots, n-1$ by the system $\left\{\frac{\overline{\partial}}{\partial p_i}\right\}$, $i = 1, \ldots, n$ taking into account relation (3.8) for some easier calculations.

3.2 Cohomology of c-indicatrix cotangent bundle

Since the vertical Liouville-Hamilton vector field C^* is orthogonal to the level hypersurfaces of the fundamental function K, the vector fields $\left\{\overline{X}^a, \xi^*, \frac{\overline{\partial}}{\partial p_a}\right\}$ are tangent to these hypersurfaces in TM^0, so they generate the distribution $\{C^*\}^\perp$ which is the orthogonal complement of $\{C^*\}$ in $T(T^*M_0)$. The vertical indicatrix distribution $\mathcal{L}_{C^*}$ is locally generated by $\left\{\frac{\overline{\partial}}{\partial p_a}\right\}$, $a = 1, \ldots, n-1$, and the vertical foliation has the structural bundle locally generated by $\left\{\frac{\overline{\partial}}{\partial p_a}, C^*\right\}$, $a = 1, \ldots, n-1$.

Also, if we consider the line distribution $\{\xi^*\}$ spanned by the horizontal Liouville-Hamilton vector field ξ^* and its complement in $H(T^*M_0)$ denoted by $\mathcal{L}_{\xi^*}$, then it is easy to see that $\mathcal{L}_{\xi^*}$ is locally spanned by the vector fields $\{\overline{X}^a\}$, $a = 1, \ldots, n-1$, and we have the decomposition

$$\{C^*\}^\perp = \{\xi^*\} \oplus \mathcal{L}_{\xi^*} \oplus \mathcal{L}_{C^*}. \tag{3.10}$$

For any $c > 0$, we consider now the *c-indicatrix cotangent bundle* over M, given by

$$I(M,K)(c) = \bigcup_{x \in M} I_x(M,K)(c) \ , \ I_x(M,K)(c) = \{p \in T^*_x M_0 \ : \ K(x,p) = c\}.$$

According to [5, 8], if $(\widetilde{N}, \widetilde{g}, \widetilde{J})$ is an (almost) Kähler manifold, where $\widetilde{g}$ is the Riemannian metric and $\widetilde{J}$ is the (almost) complex structure on $\widetilde{N}$, then N is a CR*-submanifold* of $\widetilde{N}$ if N admits two complementary orthogonal distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ such that

i) $\mathcal{D}$ is $\widetilde{J}$-invariant, i.e., $\widetilde{J}(\mathcal{D}) \subset \mathcal{D}$;

ii) $\mathcal{D}^\perp$ is $\widetilde{J}$-anti-invariant, i.e., $\widetilde{J}\left(\mathcal{D}^\perp\right) \subset (TN)^\perp$.

$\mathcal{D}$ is called *maximal complex (holomorphic)* distribution of N and $\mathcal{D}^\perp$ is called *totally real* distribution of N.

We have

Proposition 3.4. *Let $i : I(M,K)(c) \hookrightarrow T^*M_0$ be the imersion of $I(M,K)(c)$ in T^*M_0. Then $I(M,K)(c)$ is a CR-submanifold of T^*M_0 with holomorphic distribution given by $\mathcal{D} = \mathcal{L}_{\xi^*} \oplus \mathcal{L}_{C^*}$ and the totally real distribution given by $\mathcal{D}^\perp = \{\xi^*\}$.*

Proof. We have that $\{C^*\}^\perp = \{\xi^*\} \oplus \mathcal{L}_{\xi^*} \oplus \mathcal{L}_{C^*}$ is the tangent bundle of $I(M,K)(c)$. Taking into account the behaviour of the almost complex structure J of (T^*M_0, G) we have

$$J\left(\mathcal{L}_{\xi^*} \oplus \mathcal{L}_{C^*}\right) \subset \mathcal{L}_{C^*} \oplus \mathcal{L}_{\xi^*} = \mathcal{L}_{\xi^*} \oplus \mathcal{L}_{C^*} \ , \ J\left(\{\xi^*\}\right) \subset \{C^*\} = \left(\{C^*\}^\perp\right)^\perp$$

which end's the proof. $\square$

It is well known, see [12, 13], that the totally real subbundle of a CR-submanifold of an (almost) Kähler manifold is integrable and its maximal complex (holomorphic) subbundle is minimal. Then we obtain again

Proposition 3.5. *The distribution $\{\xi^*\}$ is integrable.*

and

Proposition 3.6. *The distribution $\mathcal{L}_{\xi^*} \oplus \mathcal{L}_{C^*}$ is minimal.*

We consider now $\{\omega_1, \ldots, \omega_{n-1}\}$ the dual frame of the vector fields $\left\{\overline{X}^1, \ldots, \overline{X}^{n-1}\right\}$ and $\{\theta_1, \ldots, \theta_{n-1}\}$ the dual frame of the vector fields $\left\{\frac{\overline{\partial}}{\overline{\partial} p_1}, \ldots, \frac{\overline{\partial}}{\overline{\partial} p_{n-1}}\right\}$. Then the general theory for cohomology of CR-submanifolds of an (almost) Kähler manifolds, [13], leads to

Theorem 3.1. *The differential form*

$$\nu = \omega_1 \wedge \ldots \wedge \omega_{n-1} \wedge \theta_1 \wedge \ldots \wedge \theta_{n-1}$$

is closed and it defines a cohomology class

$$[\nu] \in H^{2n-2}\left(I(M,K)(c)\right). \tag{3.11}$$

Definition 3.1. The cohomology class $[\nu]$ is called the canonical class of the c-indicatrix cotangent bundle $I(M,K)(c)$ of a Cartan space (M,K).

Remark 3.2. The form ν which defines the canonical class can be expressed in the form

$$\nu = \frac{(-1)^{n-1}}{(n-1)!}\left(i^*\Omega\right)^{n-1},$$

where Ω is the fundamental form given in (1.12).

Since $I(M,K)(c)$ is compact when M is compact, acording to [13] we have

Corollary 3.1. *If the cohomology groups $H^{2k}(I(M,K)(c)) = 0$ for some $k < n$ then either holomorphic distribution $\mathcal{L}_{\xi^*} \oplus \mathcal{L}_{C^*}$ is not integrable or its totally real distribution $\{\xi^*\}$ is not minimal.*

4 Some linear connections on a Cartan space

In this section, following some ideas from [8], we investigate the existence of some linear connections on a Cartan space, related with the vertical and Liouville-Hamilton foliations on it.

4.1 The Vrănceanu connection on $(T^*M_0, G, \mathcal{F}_V)$

First of all we have to remark that on a Cartan space (M, K) there exists the canonical metrical N-linear connection, [23]: $C\Gamma(N) = (H^i_{jk}, C_i^{jk})$. Its local coefficients, with respect to adapted basis $\{\delta_i = \frac{\delta}{\delta x^i}, \partial_i = \frac{\partial}{\partial p_i}\}$, are

$$H^i_{jk} = \frac{1}{2} g^{is} \left(\delta_j g_{sk} + \delta_k g_{js} - \delta_s g_{jk} \right), \tag{4.1}$$

and $C_i^{jk} = g_{is} C^{sjk}$. According to general discussion about the Vrănceanu connection on a foliated semi-Riemannian manifold from [8] (section 3.1), see also [10], we obtain in the case of the cotangent bundle of a Cartan space (M, K), endowed with the metric (1.10) and with vertical foliation, $(T^*M_0, G, \mathcal{F}_V)$, the following expression for the Vrănceanu connection $\check{\nabla}$:

$$\check{\nabla}_{\partial^j} \partial^i = \check{C}_k^{ij} \partial^k, \quad \check{\nabla}_{\delta_j} \partial^i = \check{D}^i_{jk} \partial^k,$$
$$\check{\nabla}_{\partial^j} \delta_i = \check{L}_i^{kj} \delta_k, \quad \check{\nabla}_{\delta_j} \delta_i = \check{F}^k_{ij} \delta_k.$$

The local coefficients of the Vrănceanu connection with respect to adapted basis $\{\delta_i, \partial_i\}$, are given by:

$$\check{C}_k^{ij} = \frac{1}{2} g_{kh} \left(\partial^j g^{hi} + \partial^i g^{hj} - \partial^h g^{ij} \right) = -g_{kh} C^{hij} = -C_k^{ij},$$
$$\check{D}^i_{jk} = -\partial^i N_{jk}, \quad \check{L}_i^{kj} = 0, \quad \check{F}^k_{ij} = H^k_{ij}. \tag{4.2}$$

The only nonzero local components of the torsion tensor field $\check{T}$ of the Vrănceanu connection are

$$\check{T}(\delta_i, \delta_j) = \check{T}_{ijk} \partial^k = (\delta_j N_{ik} - \delta_j N_{jk}) \partial^k.$$

Hence, it results the following:

Proposition 4.1. *The Vrănceanu connection $\check{\nabla}$ on $(T^*M_0, G, \mathcal{F}_V)$ is torsion-free if and only if the canonical nonlinear connection N is integrable.*

Remark 4.1. In [8] (Remark 1.1, p.99) it is mentioned that the Vrănceanu connection on a foliated manifold is exactly the Vaisman connection (or second connection) on that manifold. Related to Vaisman connection there is the notion of *Reinhart space*. A Riemannian foliated manifold (M, g, F) is called a Reinhart space iff

$$(\check{\nabla}_X g)(Y, Z) = 0, \quad \forall X \in \Gamma(TF), Y, Z \in \Gamma(T^\perp F).$$

Proposition 4.2. *Let (M, K) be a Cartan space. The foliated manifold $(T^*M_0, G, \mathcal{F}_V)$ is a Reinhart space if and only if the metric $g = (g^{ij})$ is Riemannian on M.*

Proof. Since $\check{\nabla}_{\partial^j} \delta_i = 0$, we have

$$(\check{\nabla}_{\partial^k} G)(\delta_i, \delta_j) = \partial^k g_{ij},$$

which vanishes iff g_{ij} does not depends by the momenta p_k, for all indices i, j, k. It follows that the condition $(\check{\nabla}_X G)(Y, Z) = 0, \quad \forall X \in \Gamma(V(T^*M_0)), Y, Z \in \Gamma(H(T^*M_0))$ is equivalent to $g^{ij} = g^{ij}(x)$, so it is a Riemannian metric on M. $\square$

4.2 The Vaisman connection on $V(T^*M_0)$

Taking into account the decomposition (3.1) and the adapted basis to this decomposition, we shall construct an anlogous of the Vaisman connection on a Riemannian foliated manifold with respect to vertical Hamilton-Liouville foliation $\mathcal{L}_{C^*}$.

Let (M,g) be a Riemannian foliated manifold, with D', D, the structural and transversal distributions, respectively. The *Vaisman connection* on M is a connection ∇^v on TM uniquely defined by the following conditions, [25]:

a) If $Y\in D'$ (respectively $\in D$), then $\nabla^v_X Y\in D'$ (respectively $\in D$), for every vector field X.

b) $v(T^v(X,Y))=0$, (respectively $h(T^v(X,Y))=0$) if at least one of the arguments is in D', respectively in D, where v,h are the projection morphisms of TM on D' and D, respectively.

c) The induced connections on D' and D are metric connections.

We return now to the Cartan space (M,K), with metric G on T^*M given by (1.10).

Let $v:V(T^*M_0)\to\mathcal{L}_{C^*}$, $h:V(T^*M_0)\to\{C^*\}$ be the projection morphisms. We are looking for a connection on $V(T^*M_0)$ with the same properties as the Vaisman connection:

a) If $Y\in\mathcal{L}_{C^*}$ (respectively $\in\{C^*\}$), then $\nabla^v_X Y\in\mathcal{L}_{C^*}$ (respectively $\in\{C^*\}$), for every vertical vector field X.

b) $v(T^v(X,Y))=0$, (respectively $h(T^v(X,Y))=0$) if at least one of the arguments is in $\mathcal{L}_{C^*}$, respectively in $\{C^*\}$, where T^v is the torsion of the connection ∇^v.

c) For every $X,Y,Y'\in\mathcal{L}_{C^*}$ (respectively $\in\{C^*\}$),

$$(\nabla^v_X G)(Y,Y')=0. \tag{4.3}$$

d) Moreover, we need to give $\nabla^v_X Y$ for every $X\in H(T^*M_0)$ and we put the above a), b) conditions also for X an horizontal vector field.

In the following the indices take the values $i,j,...=\overline{1,n}$ and $a,b,...,=\overline{1,n-1}$. We request the above conditions on the adapted basis

$$\left\{\frac{\overline{\partial}}{\overline{\partial}p_1},\dots,\frac{\overline{\partial}}{\overline{\partial}p_{n-1}},C^*\right\} \tag{4.4}$$

on $V(T^*M_0)$. The Riemannian metric induced by G on $V(T^*M_0)$ has the matrix

$$h=\begin{pmatrix} (h^{ab})_{n-1\times n-1} & O_{n-1\times 1}\\ O_{1\times n-1} & K^2\end{pmatrix},\quad h^{ab}=G(\overline{\partial}^a,\overline{\partial}^b)=g^{ab}-K^2t^at^b, \tag{4.5}$$

with respect to the adapted basis, where we have denoted $\frac{\overline{\partial}}{\overline{\partial}p^b}=\overline{\partial}^b$.

Let us consider the local expression of ∇^v:

$$\nabla^v_{\overline{\partial}^b}\overline{\partial}^a=s^{ab}_c\overline{\partial}^c,\quad \nabla^v_{C^*}\overline{\partial}^a=s^a_c\overline{\partial}^c,\quad \nabla^v_{\overline{\partial}^a}C^*=s^aC^*,\quad \nabla^v_{C^*}C^*=sC^*, \tag{4.6}$$

$$\nabla^v_{\delta_i}\overline{\partial}^a=\beta^a_{bi}\overline{\partial}^b,\quad \nabla^v_{\delta_i}C^*=\beta_iC^*.$$

Taking into account Proposition 3.2, we compute

$$T^v(\overline{\partial}^a, \overline{\partial}^b) = (s_c^{ab} - s_c^{ba})\overline{\partial}^c - t^a\overline{\partial}^b + t^b\overline{\partial}^a \in \mathcal{L}_{C^*}, \tag{4.7}$$

and by condition b) it results

$$s_c^{ab} = s_c^{ba}, \quad \forall\, c \neq a, b, \quad s_b^{ba} = s_b^{ab} + t^a, \tag{4.8}$$

for all $a, b, c = \overline{1, n-1}$.

We also have

$$T^v(\overline{\partial}^a, C^*) = s^a C^* - s_b^a\overline{\partial}^b - \overline{\partial}^a = 0,$$

hence

$$s^a = 0, \quad s_b^a = 0, \quad \forall\, b \neq a, \quad s_a^a = -1, \quad \forall a = \overline{1, n-1}. \tag{4.9}$$

Now we have

$$\nabla^v_{\overline{\partial}^a} C^* = 0; \quad \nabla^v_{C^*}\overline{\partial}^a = -\overline{\partial}^a, \quad \forall a = \overline{1, n-1}. \tag{4.10}$$

By conditions c), $(\nabla^v_{C^*}G)(C^*, C^*) = 0$ implies $C^*(K^2) = 2sK^2$ and so $s = 1$, since $C^*(K^2) = p_i\frac{\partial}{\partial p_i}(p_j p_k g^{jk}) = 2K^2$, by (1.7). Also by $(\nabla^v_{\overline{\partial}^a}G)(\overline{\partial}^b, \overline{\partial}^c) = 0$ we have

$$\overline{\partial}^a(G(\overline{\partial}^b, \overline{\partial}^c)) - G(\nabla^v_{\overline{\partial}^a}\overline{\partial}^b, \overline{\partial}^c) - G(\nabla^v_{\overline{\partial}^a}\overline{\partial}^c, \overline{\partial}^b) = 0,$$

for all $a, b, c = \overline{1, n-1}$ and $\overline{\partial}^a(G(\overline{\partial}^b, \overline{\partial}^c)) = \overline{\partial}^a(g^{bc}) - t^c h^{ba} + t^b h^{ca}$, where h^{ab} are given by (4.5). Using the same method while we determined the Christoffel's symbols, we obtain:

$$2s_d^{ba}h^{dc} = \overline{\partial}^a(g^{bc}) - 2t^b h^{ca}. \tag{4.11}$$

Let $(h_{ij})_{n\times n}$ be the inverse of the matrix h from (4.5). Finally, we have

$$s_d^{ba} = \frac{1}{2}h_{dc}\overline{\partial}^a(g^{bc}), \quad \forall\, d \neq a, \quad s_a^{ba} = \frac{1}{2}h_{ac}\overline{\partial}^a(g^{bc}) - t^b. \tag{4.12}$$

The conditions d) give us:

$$\beta_i = 0, \quad \beta^a_{bi}\overline{\partial}^b = \overline{\partial}^a(N_{ij})\partial^j. \tag{4.13}$$

Proposition 4.3. *The local coefficients of the connection ∇^v on $V(T^*M_0)$ defined by the conditions a), b), c), d) above described are given with respect to the adapted basis* (4.4) *by the relations* (4.9), (4.12), (4.13) *and $\nabla^v_{C^*}C^* = C^*$.*

Remark 4.2. The Vrănceanu connection $\check{\nabla}$ satisfies

$$\check{\nabla}_{\overline{\partial}^a} C^* = \overline{\partial}^a,$$

so by (4.10), the restriction of the Vrănceanu connection to the vertical bundle $V(T^*M)$ is diferent by the Vaisman connection above determined.

Now, for every fixed point $x_0 \in M$, the leaf $T^*_{x_0}M$ of the vertical foliation $\mathcal{F}_V$ is also a Riemannian manifold, foliated by $\mathcal{L}_{C^*}$. The Riemannian metric is

$$h_{x_0} = \begin{pmatrix} (h^{ab}(x_0,p))_{n-1\times n-1} & O_{n-1\times 1} \\ O_{1\times n-1} & K^2(x_0,p) \end{pmatrix}.$$

The connection ∇^v is exactly the Vaisman connection on $(T^*_{x_0}M, h_{x_0}, \mathcal{L}_{C^*})$ and we have:

Proposition 4.4. *For every fixed point $x_0 \in M$, $(T^*_{x_0}M, h_{x_0}, \mathcal{L}_{C^*})$ is a Reinhart space.*

Proof. Indeed, we can calculate

$$(\nabla^v_{\overline{\partial}^a} G)(C^*, C^*) = \overline{\partial}^a(K^2) - 2G(\nabla^v_{\overline{\partial}^a} C^*, C^*) = \overline{\partial}^a(K^2) = \frac{\partial K^2}{\partial p_a} - t^a C^*(K^2) = 0,$$

for every $a = 1, \ldots, n-1$. □

5 Subfoliations in the cotangent manifold of a Cartan space

In this section, following [15], we briefly recall the notion of a (q_1, q_2)-codimensional subfoliation on a manifold and we identify a $(n, 2n-1)$-codimensional subfoliation $(\mathcal{F}_V, \mathcal{F}_{C^*})$ on the cotangent manifold T^*M_0 of a Cartan space (M, K), where $\mathcal{F}_V$ is the vertical foliation and $\mathcal{F}_{C^*}$ is the line foliation spanned by the vertical Liouville-Hamilton vector field C^*. Firstly we make a general approach about basic connections on the normal bundles related to this subfoliation and next a triple of adapted basic connections with respect to this subfoliation is given.

Definition 5.1. Let M be a n-dimensional manifold and TM its tangent bundle. A (q_1, q_2)-*codimensional subfoliation* on M is a couple (F_1, F_2) of integrable subbundles F_k of TM of dimension $n - q_k$, $k = 1, 2$ and F_2 being at the same time a subbundle of F_1.

For a subfoliation (F_1, F_2), its normal bundle is defined as $Q(F_1, F_2) = QF_{21} \oplus QF_1$, where QF_{21} is the quotient bundle F_1/F_2 and QF_1 is the usual normal bundle of F_1. So, an exact sequence of vector bundles

$$0 \longrightarrow QF_{21} \xrightarrow{i} QF_2 \xrightarrow{\pi} QF_1 \longrightarrow 0 \tag{5.1}$$

appears in a canonical way.

Also if we consider the canonical exact sequence associated to the foliation given by an integrable subbundle F, namely

$$0 \longrightarrow F \xrightarrow{i_F} TM \xrightarrow{\pi_F} QF \longrightarrow 0$$

then we recall that a connection $\nabla : \Gamma(TM) \times \Gamma(QF) \to \Gamma(QF)$ on the normal bundle QF is said to be *basic* if

$$\nabla_X Y = \pi_F[X, \widetilde{Y}] \tag{5.2}$$

for any $X \in \Gamma(F)$, $\widetilde{Y} \in \Gamma(TM)$ such that $\pi_F(\widetilde{Y}) = Y$.

Similarly, for a (q_1, q_2)-subfoliation (F_1, F_2) we can consider the following exact sequence of vector bundles

$$0 \longrightarrow F_2 \xrightarrow{i_0} F_1 \xrightarrow{\pi_0} QF_{21} \longrightarrow 0 \tag{5.3}$$

and according to [15] a connection ∇ on QF_{21} is said to be basic with respect to the subfoliation (F_1, F_2) if

$$\nabla_X Y = \pi_0[X, \widetilde{Y}] \tag{5.4}$$

for any $X \in \Gamma(F_2)$ and $\widetilde{Y} \in \Gamma(F_1)$ such that $\pi_0(\widetilde{Y}) = Y$.

5.1 A $(n, 2n-1)$-codimensional subfoliation $(\mathcal{F}_V, \mathcal{F}_{C^*})$ of (T^*M_0, G)

Taking into account the discussion from the previous section, for a n-dimensional Cartan space (M, K), we have on the $2n$-dimensional cotangent manifold T^*M_0 a $(n, 2n-1)$-codimensional foliation $(\mathcal{F}_V, \mathcal{F}_{C^*})$. We also notice that the metric structure G on T^*M_0 given by (1.10) is compatible with the subfoliated structure, that is

$$Q\mathcal{F}_V \cong H(T^*M_0),\ Q\mathcal{F}_{C^*} \cong \{C^*\}^\perp,\ V(T^*M_0)/\{C^*\} \cong \mathcal{L}_{C^*}.$$

Let us consider the following exact sequences associated to the subfoliation $(\mathcal{F}_V, \mathcal{F}_{C^*})$

$$0 \longrightarrow \{C^*\} \xrightarrow{i_0} V(T^*M_0) \xrightarrow{\pi_0} \mathcal{L}_{C^*} \longrightarrow 0,$$

and to foliations $\mathcal{F}_V$ and $\mathcal{F}_{C^*}$, respectively

$$0 \longrightarrow V(T^*M_0) \xrightarrow{i_1} T(T^*M_0) \xrightarrow{\pi_1} H(T^*M_0) \longrightarrow 0,$$

$$0 \longrightarrow \{C^*\} \xrightarrow{i_2} T(T^*M_0) \xrightarrow{\pi_2} \{C^*\}^\perp \longrightarrow 0,$$

where i_0, i_1, i_2, π_0, π_1, π_2 are the canonical inclusions and projections, respectively.

A triple $(\nabla^1, \nabla^2, \nabla)$ of basic connections on normal bundles $\mathcal{L}_{C^*}$, $H(T^*M_0)$, $\{C^*\}^\perp$, respectively, is called in [15] *adapted* to the subfoliation $(\mathcal{F}_V, \mathcal{F}_{C^*})$.

Our goal is to determine such a triple of connections, adapted to this subfoliation.

By (5.4) a connection ∇^1 on $\mathcal{L}_{C^*}$ is basic with respect to the subfoliation $(\mathcal{F}_V, \mathcal{F}_{C^*})$ if

$$\nabla^1_X Z = \pi_0[X, \widetilde{Z}],\ \forall\, X \in \Gamma(\{C^*\}),\ \forall\, \widetilde{Z} \in \Gamma(V(T^*M_0)),\ \pi_0(\widetilde{Z}) = Z. \tag{5.5}$$

Proposition 5.1. *A connection ∇^1 on $\mathcal{L}_{C^*}$ is basic if and only if*

$$\nabla^1_{C^*} Z = [C^*, Z],\ \forall\, Z \in \Gamma(\mathcal{L}_{C^*}).$$

Proof. Let $\nabla^1 : V(T^*M_0)\times\mathcal{L}_{C^*} \to \mathcal{L}_{C^*}$ be a connection on $\mathcal{L}_{C^*}$ such that $\nabla^1_{C^*}Z = [C^*, Z]$. Let $X \in \Gamma(\{C^*\})$ be a section in the structural bundle of a the line foliation $\mathcal{F}_{C^*}$, so its form is $X = aC^*$, with a a differentiable function on T^*M_0. An arbitrary vertical vector field $\widetilde{Z}$ which projects into $Z \in \mathcal{L}_{C^*}$ is in the form

$$\widetilde{Z} = Z + bC^*$$

with b a differentiable function on T^*M_0.

We have

$$\begin{aligned} [X, \widetilde{Z}] &= [aC^*, Z + bC^*] \\ &= a[C^*, Z] + (aC^*(b) - bC^*(a) - Z(a))C^*. \end{aligned}$$

According to the second relation from (3.7) for any $Z = Z_i\overline{\partial}^i \in \Gamma\left(\mathcal{L}_{C^*}\right)$, we have

$$[C^*, Z] = (C^*(Z_i) - Z_i)\overline{\partial}^i \in \Gamma\left(\mathcal{L}_{C^*}\right),$$

so $\pi_0[X, \widetilde{Z}] = a[C^*, Z]$. We also have $\nabla^1_X Z = a\nabla^1_{C^*}Z = a[C^*, Z] = \pi_0[X, \widetilde{Z}]$, hence ∇^1 is a basic connection on $\mathcal{L}_{C^*}$.

Conversely, by the second relation from (3.7), in the adapted basis $\left\{\overline{\partial}^i, C^*\right\}$ in $V(T^*M_0)$, every basic connection ∇^1 on $\mathcal{L}_{C^*}$ is locally satisfying

$$\nabla^1_{C^*}\overline{\partial}^i = -\overline{\partial}^i, \tag{5.6}$$

for any $i = 1, \ldots, n$, where we have used the Remark 3.1.

Now, if (5.6) is satsfied, then

$$\nabla^1_{C^*}Z = C^*(Z_i)\overline{\partial}^i + Z_i\nabla^1_{C^*}\overline{\partial}^i = C^*(Z_i)\overline{\partial}^i - Z_i\overline{\partial}^i.$$

Hence the condition (5.6) is equivalent with $\nabla^1_{C^*}Z = [C^*, Z], \quad \forall Z \in \Gamma\left(\mathcal{L}_{C^*}\right)$. □

Moreover, by relation (3.5), it follows that condition (5.6) has geometrical meaning. We obtain the locally characterisation:

Proposition 5.2. *A connection ∇^1 on $\mathcal{L}_{C^*}$ is basic if and only if in an adapted local chart the relation* (5.6) *holds.*

Now, by (5.2), a connection ∇^2 on $H(T^*M_0)$ is basic with respect to the vertical foliation $\mathcal{F}_V$ if

$$\nabla^2_X Y = \pi_1[X, \widetilde{Y}] \tag{5.7}$$

for any $X \in \Gamma(V(T^*M_0))$ and $\widetilde{Y} \in \Gamma(T(T^*M_0))$ such that $\pi_1(\widetilde{Y}) = Y$.

Proposition 5.3. *A connection ∇^2 on $H(T^*M_0)$ is basic if and only if in an adapted local frame $\{\delta_i, \partial^i\}$ on $T(T^*M_0)$ we have*

$$\nabla^2_{\partial^j}\delta_i = 0$$

for any $i, j = 1, \ldots, n$.

Proof. Obviously, the above condition has geometrical meaning since if $(\widetilde{U}, (\widetilde{x}^{i_1}, \widetilde{p}_{i_1}))$ is another local chart on T^*M_0, in $U \cap \widetilde{U} \neq \emptyset$, we have

$$\frac{\delta}{\delta \widetilde{x}^{i_1}} = \frac{\partial x^i}{\partial \widetilde{x}^{i_1}} \frac{\delta}{\delta x^i}, \quad \frac{\partial}{\partial \widetilde{p}_{j_1}} = \frac{\partial \widetilde{x}^{j_1}}{\partial x^j} \frac{\partial}{\partial p_j}.$$

If ∇^2 is a basic connection with respect to the vertical foliation, then by definition it results

$$\nabla^2_{\partial^j}\delta_i = \pi_1 \left[\delta_i, \partial^j\right] = \pi_1 \left(-\frac{\partial N_{ki}}{\partial p_j} \partial^k \right) = 0.$$

Conversely, let $\nabla^2 : T(T^*M_0) \times H(T^*M_0) \to H(T^*M_0)$ be a connection on $H(T^*M_0)$ which locally satisfies

$$\nabla^2_{\partial^j}\delta_i = 0$$

for any $i, j = 1, \ldots, n$.

An arbitrary vertical vector field X has local expression $X = X_i \partial^i$ and a vector field $\widetilde{Y}$ whose horizontal projection is $Y = Y_h^i \delta_i$ is by the form $\widetilde{Y} = Y + Y_i^v \partial^i$.

We calculate

$$\nabla^2_X Y = X_i \nabla^2_{\partial^i} \left(Y_h^j \delta_j \right) = X_i \frac{\partial Y_h^j}{\partial p_i} \delta_j,$$

$$\left[X, \widetilde{Y}\right] = X_i \frac{\partial Y_h^j}{\partial p_i} \delta_j + \left(X_i \frac{\partial Y_v^j}{\partial p_i} - Y_h^i \frac{\delta X_j}{\delta x^i} \right) \partial^j + X_i Y_h^j \left[\partial^i, \delta_j\right],$$

hence the relation (5.7) is verified, since $\left[\partial^i, \delta_j\right] \in \Gamma(V(T^*M_0))$. So, ∇^2 is a basic connection with respect to the vertical foliation $\mathcal{F}_V$. $\square$

Also, by (5.2), a connection ∇ on $\{C^*\}^\perp$ is basic with respect to the line foliation $\mathcal{F}_{C^*}$ if

$$\nabla_X Y = \pi_2[X, \widetilde{Y}] \tag{5.8}$$

for any $X \in \Gamma(\{C^*\})$ and $\widetilde{Y} \in \Gamma(T(T^*M_0))$ such that $\pi_2(\widetilde{Y}) = Y$.

We have the following locally characterisation of a basic connection on $\{C^*\}^\perp$:

Proposition 5.4. *A connection ∇ on $\{C^*\}^\perp$ is basic with respect to the line foliation $\mathcal{F}_{C^*}$ if and only if in an adapted local frame $\left\{\delta_i, \overline{\partial}^i\right\}$ on $\{C^*\}^\perp$ we have*

$$\nabla_{C^*}\delta_i = 0, \quad \nabla_{C^*}\overline{\partial}^i = -\overline{\partial}^i, \tag{5.9}$$

for any $i = 1, \ldots, n$.

Proof. Let ∇ be a basic connection on $\{C^*\}^\perp$. Since $\{C^*\}^\perp$ is locally generated by $\left\{\delta_i, \overline{\partial}^i\right\}$, the condition (5.8) give us the following relations:

$$\nabla_{C^*}\delta_i = \pi_2\left[C^*, \delta_i + fC^*\right] = \pi_2\left((-N_{ji} + C^*(N_{ji}))\partial^j + C^*(f)C^*\right) = 0,$$

since $C^*(N_{ji}) = N_{ji}$, by the homogeneity of degree 1 in mementa of functions N_{ji}, see [23], and

$$\nabla_{C^*}\overline{\partial}^i = \pi_2\left[C^*, \overline{\partial}^i + fC^*\right] = \pi_2\left(-\overline{\partial}^i + C^*(f)C^*\right) = -\overline{\partial}^i.$$

Conversely, let us consider $\nabla : T(T^*M_0) \times \{C^*\}^\perp \to \{C^*\}^\perp$ be a connection on $\{C^*\}^\perp$ which locally satisfies (5.9).

An arbitrary vector field $Y \in \Gamma\left(\{C^*\}^\perp\right)$ is locally given by $Y = Y_h^i\delta_i + Y_i\overline{\partial}^i$ and a vector field $\widetilde{Y} \in \Gamma(T(T^*M_0))$ which projects by π_2 in Y is $\widetilde{Y} = fC^* + Y$. For an arbitrary vector field $X = aC^* \in \Gamma(\{C^*\})$, we calculate

$$\begin{aligned}
\nabla_X Y &= \nabla_{aC^*}(Y_h^i\delta_i + Y_i\overline{\partial}^i)\\
&= aC^*(Y_h^i)\delta_i + aY_h^i\nabla_{C^*}\delta_i + aC^*(Y_i)\overline{\partial}^i + aY_i\nabla_{C^*}\overline{\partial}^i\\
&= aC^*(Y_h^i)\delta_i + a(C^*(Y_i) - Y_i)\overline{\partial}^i,
\end{aligned}$$

and

$$\begin{aligned}
\pi_2[X, \widetilde{Y}] &= \pi_2\left(aC^*(f)C^* - fC^*(a)C^* + aC^*(Y_h^i)\delta_i + aY_h^i\left[C^*, \delta_i\right]\right)\\
&\quad + \pi_2\left(a(C^*(Y_i) - Y_i)\overline{\partial}^i - Y_h^i\delta_i C^* - Y_i\frac{\overline{\partial}a}{\overline{\partial}p_i}C^*\right)\\
&= aC^*(Y_h^i)\delta_i + a(C^*(Y_i) - Y_i)\overline{\partial}^i,
\end{aligned}$$

since $[C^*, \delta_i] = 0$. Thus, we have obtained that the connection ∇ is a basic one. $\square$

5.2 A triple of adapted basic connections to subfoliation $(\mathcal{F}_V, \mathcal{F}_{C^*})$

The restriction of Vrănceanu connection introduced in subsection 4.1, to $T(T^*M_0) \times H(T^*M_0)$, is a connection on $H(T^*M_0)$ denoted by $\check{\nabla}_1$, which satisfies the conditions from Proposition 5.3, (see relations (4.2)), so it is a basic connection on $H(T^*M_0)$ with respect to vertical foliation $\mathcal{F}_V$.

The Vaisman connection from subsection 4.2 induces a connection on $\mathcal{L}_{C^*}$, which satisfies the conditions from Proposition 5.2, (see relations (4.10)), so it is a basic connection on $\mathcal{L}_{C^*}$ with respect to Liouville vertical foliation $\mathcal{F}_{C^*}$.

Hence we have the basic connection $\check{\nabla}_1$ on $H(T^*M_0)$ and the basic connection ∇^v on $\mathcal{L}_{C^*}$. Following [15], we can build now a connection on $\{C^*\}^\perp$ as follows:

$$\overline{\nabla} : T(T^*M_0) \times \{C^*\}^\perp \to \{C^*\}^\perp\,,\; \overline{\nabla}_X Z = \check{\nabla}_{1X} Z^h + \nabla^v_X Z'$$

for any $Z = Z^h + Z' \in \Gamma\left(\{C^*\}^\perp\right) = \Gamma\left(H(T^*M_0)\right) \oplus \Gamma\left(\mathcal{L}_{C^*}\right)$ and $X \in \Gamma(T(T^*M_0))$.

By direct calculus we have

$$\overline{\nabla}_{C^*}\delta_i = \check{\nabla}_{1C^*}\delta_i = 0, \quad \overline{\nabla}_{C^*}\overline{\partial}^i = \nabla^v_{C^*}\overline{\partial}^i = -\overline{\partial}^i,$$

so the connection $\overline{\nabla}$ satisfies conditions from Proposition 5.4, hence it is a basic connection on $\{C^*\}^\perp$.

Now, for the $(n, 2n-1)$-codimensional subfoliation $(\mathcal{F}_V, \mathcal{F}_{C^*})$ we can consider the exact sequence (5.1), namely

$$0 \longrightarrow \mathcal{L}_{C^*} \xrightarrow{i} \{C^*\}^\perp \xrightarrow{\pi} H(T^*M_0) \longrightarrow 0$$

and the general theory of (q_1, q_2)-codimensional foliations, see [15], leads to the following results:

Proposition 5.5. *For the triple basic connections $(\nabla^v, \check{\nabla}_1, \overline{\nabla})$ on $\mathcal{L}_{C^*}$, $H(T^*M_0)$ and $\{C^*\}^\perp$, respectively, we have*

$$i\left(\nabla^v_X Y\right) = \overline{\nabla}_X i(Y)\,,\ \pi\left(\overline{\nabla}_X Z\right) = \check{\nabla}_{1X}\pi(Z)\,,\ \forall\, Y \in \Gamma\left(\mathcal{L}_{C^*}\right),\ Z \in \Gamma\left(\{C^*\}^\perp\right).$$

Proposition 5.6. *The triple of basic connections $(\nabla^v, \check{\nabla}_1, \overline{\nabla})$ is adapted to the subfoliation $(\mathcal{F}_V, \mathcal{F}_{C^*})$ of (T^*M_0, G).*

Remark 5.1. We notice that the connection $\nabla = \nabla^v \oplus \check{\nabla}_1$ is a basic connection on $Q(\mathcal{F}_V, \mathcal{F}_{C^*}) = \mathcal{L}_{C^*} \oplus H(T^*M_0) = \{C^*\}^\perp$ and its curvature K satisfy

$$K(X, Y) = 0\,,\ \forall\, X, Y \in \Gamma(\{C^*\}). \tag{5.10}$$

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Cristian Ida and Adelina Manea
Department of Mathematics and Computer Science
University Transilvania of Braşov
Address: Braşov 500091, Str. Iuliu Maniu 50, România
email: *cristian.ida@unitbv.ro; amanea28@yahoo.com*